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Formulae for null curves deriving from elliptic curves

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Abstract

Any elliptic curve can be realised in the tangent bundle of the complex projective line as a double cover branched at four distinct points on the zero section. Such a curve generates, via classical osculation duality, a null curve in $\mathbb{C}^3$ and thus an algebraic minimal surface in $\mathbb{R}^3$. We derive simple formulae for the coordinate functions of such a null curve.

MSC: primary 53A10; secondary 53A05; 14Q05

1. Introduction

Suppose that (g, f) is a pair of holomorphic functions on a Riemann surface M. The following are versions of the classical Weierstrass formulae in free form [1,3,5,9,10], and give the coordinate functions of a *null* holomorphic curve $\psi : M^* \longrightarrow \mathbb{C}^3$; which is to say that ψ satisfies $(\psi_1')^2 + (\psi_2')^2 + (\psi_3')^2 = 0$. Here M^* is obtained from M by deleting a finite number of points, and $\mathrm{d}f/\mathrm{d}g = f'/g'$, $\mathrm{d}^2 f/\mathrm{d}g^2 = (\mathrm{d}f/\mathrm{d}g)'/g'$, etc.

$$\psi_1 = -\frac{1}{2}\left\{\frac{1}{2}(1-g^2)\frac{\mathrm{d}^2 f}{\mathrm{d}g^2} + g\frac{\mathrm{d}f}{\mathrm{d}g} - f\right\} \tag{1}$$

$$\psi_2 = -\frac{i}{2}\left\{\frac{1}{2}(1+g^2)\frac{\mathrm{d}^2 f}{\mathrm{d}g^2} - g\frac{\mathrm{d}f}{\mathrm{d}g} + f\right\} \tag{2}$$

$$\psi_3 = \frac{1}{2}\left\{g\frac{\mathrm{d}^2 f}{\mathrm{d}g^2} - \frac{\mathrm{d}f}{\mathrm{d}g}\right\}. \tag{3}$$

The normalisation agrees with [5]. Every non-planar null curve in $\mathbb{C}^3$ may be described in this way; this is important in the study of minimal surfaces in $\mathbb{R}^3$, because any minimal surface in $\mathbb{R}^3$ can be described as the real part of a null curve in $\mathbb{C}^3$.

Recall that charge 2 monopole spectral curves are real elliptic curves, satisfying a certain transcendental constraint, and are explicitly exhibited as double covers of $\mathbb{P}_1$ [6]. In [11], formulae for the null curves that derive, via (1)–(3),

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from these elliptic curves are given. It turns out that their surprisingly simple form is not a result of the special nature of spectral curves; similar formulae hold in general, for double covers of the complex projective line $\mathbb{P}_1$, that are branched over four distinct points. This can be described quite simply in terms of elliptic functions, as follows in Section 2. In Section 3 we make some comments about the geometry that underlies these formulae.

2. Formulae

To fix notation, suppose that Λ is a lattice in $\mathbb{C}$, with a pair of basic periods $\{\omega_1, \omega_2\}$; let $E = \mathbb{C}/\Lambda$ and $\wp : E \longrightarrow \mathbb{P}_1$ be the auxiliary Weierstrass $\wp$-function. Furthermore let g_2 and g_3 be the invariants of $\wp$; $\wp(\omega_j/2) = e_j$, $j = 1, 2, 3$, and let $\varepsilon_1 = \sqrt{e_3 - e_2}$, $\varepsilon_2 = \sqrt{e_1 - e_3}$, $\varepsilon_3 = \sqrt{e_2 - e_1}$.

Consider the pair of meromorphic functions

$$g(u) = -\frac{1}{\varepsilon_1\varepsilon_2}(\wp(u) - e_3), \tag{4}$$

$$f(u) = \mathrm{i}\varepsilon_3\wp'(u). \tag{5}$$

Let $E^* = E \setminus \{\frac{1}{2} - \text{periods}\}$. Substituting (4) and (5) into (1)–(3) yields the null meromorphic curve $\Psi : E^* \longrightarrow \mathbb{C}^3$, with coordinate functions

$$\Psi_1(u) = -\mathrm{i}\varepsilon_3\left\{\frac{4\wp(u)^3 - 24e_3\wp(u)^2 - 2(g_2 + 24e_1e_2)\wp(u) + 17g_3 - 2e_3g_2}{16\wp'(u)}\right\}$$

$$\Psi_2(u) = -\varepsilon_3\left\{\frac{4\wp(u)^3 - 24e_3\wp(u)^2 + 2(5g_2 + 12e_1e_2)\wp(u) + 17g_3 - 2e_3g_2}{16\wp'(u)}\right\}$$

$$\Psi_3(u) = \mathrm{i}\varepsilon_1\varepsilon_2\varepsilon_3\left\{\frac{12\wp(u)^2 + 12e_3\wp(u) + g_2}{8\wp'(u)}\right\}.$$

Now let $f_j(u)$ denote the square root of $\wp(u) - e_j$, $j = 1, 2, 3$, whose residue at the origin is 1. Let $\Omega : E^* \longrightarrow \mathbb{C}^3$ be the null meromorphic curve whose coordinate functions are given by

$$\Omega(u) = (\varepsilon_1 f_1^3(2u), \varepsilon_2 f_2^3(2u), \varepsilon_3 f_3^3(2u)). \tag{6}$$

Theorem 1. *$\Psi = A\Omega$, where $A \in SO(3, \mathbb{C})$, is given by*

$$A = \frac{1}{\mathrm{i}\varepsilon_3}\begin{pmatrix} \varepsilon_1 & \varepsilon_2 & 0 \\ 0 & 0 & \mathrm{i}\varepsilon_3 \\ \varepsilon_2 & -\varepsilon_1 & 0 \end{pmatrix}. \tag{7}$$

Proof. Recall the following classical identities:

$$f_1(2u) = \frac{(e_1 - e_2)(e_1 - e_3) - (\wp(u) - e_1)^2}{\wp'(u)} \tag{8}$$

$$f_2(2u) = \frac{(e_2 - e_1)(e_2 - e_3) - (\wp(u) - e_2)^2}{\wp'(u)} \tag{9}$$

$$f_3(2u) = \frac{(e_3 - e_1)(e_3 - e_2) - (\wp(u) - e_3)^2}{\wp'(u)}. \tag{10}$$

Now substitute into (6) and apply A. □

Remark. (10) appears (with a typo) in Section 333 of [12].

3. Comments on the geometry

Recall that any complex torus $\mathbb{C}/\Lambda$ may be realised as a non-singular plane cubic, or alternatively, as the intersection of two quadrics in $\mathbb{P}_3$. The former description is related to the Weierstrass $\wp$-function, whilst in the latter the torus is embedded via theta functions [8]. We will now briefly explain how the geometry underlying the Weierstrass formulae (1)–(3) is closely connected to the latter construction and thus perhaps the existence of Ω in (6), together with its ubiquity (cf. Corollary 1) and the transparency of its description are not perhaps surprising after all. See [3,10] for the basics. In Corollary 1 we reformulate Theorem 1 in more geometric language.

Let $\mathbb{T}$ be the total space of the holomorphic tangent bundle of $\mathbb{P}_1$. $\mathbb{T}$ embeds into $\mathbb{P}_3$ where, through the addition of a single point at infinity, it is compactified to a quadric cone $\mathcal{C}(Q_1)$, with vertex v_∞ say.

Recall that classical osculation duality gives a correspondence between non-planar curves in $\mathbb{P}_3$ and $\mathbb{P}_3^*$. $\mathcal{C}(Q_1)$ determines a 'dual' quadric cone $\mathcal{C}(Q_1^*) \subset \mathbb{P}_3^*$, where Q_1^* lies on v^*, which parameterises all the hyperplanes of $\mathbb{P}_3$ lying tangent to Q_1.

Osculation of a non-planar curve $S \subset \mathbb{P}_3$, that happens to lie on $\mathcal{C}(Q_1)$, gives a curve $S^\natural \subset \mathbb{P}_3^*$, which is such that every hyperplane in $\mathbb{P}_3^*$ osculating $S^\natural$ lies tangent to the 'quadric at infinity' Q_1^*. Writing $\mathbb{C}^3 = \mathbb{P}_3^* \setminus v^*$, this means that the affine part of $S^\natural$ in $\mathbb{C}^3$ is a null curve relative to the affine cone $C(Q_1^*) \subset \mathbb{C}^3$. Every non-planar null curve in $\mathbb{C}^3$ arises in this way. This was discovered by Lie [3]; but see also [5], where it is described in the following way.

Hyperplanes of $\mathbb{P}_3$, which do not pass through v_∞, cut out global sections on $\mathbb{T}$ when embedded as above. The 2-jet of a germ of a local holomorphic section of $\mathbb{T}$ determines a global section; thus a holomorphic curve S lying in $\mathbb{T}$ generates, via 'osculation', a holomorphic curve in $\mathrm{H}^0(\mathbb{P}_1, \mathcal{O}(\mathbb{T})) \cong \mathbb{C}^3$. Those global sections that have a double root on $\mathbb{P}_1$ comprise an affine null cone in $\mathrm{H}^0(\mathbb{P}_1, \mathcal{O}(\mathbb{T}))$, which in suitable coordinates is given by $(z_1^2+z_2^2+z_3^2=0)$. If $\psi : S^* \longrightarrow \mathbb{C}^3$ is given by osculation then it is null.

Let ζ be an affine coordinate on $\mathbb{P}_1$, giving the local coordinates $(\zeta, \eta) \longrightarrow (\zeta, \eta \mathrm{d}/\mathrm{d}\zeta)$ on $\mathbb{T}$. If the affine part of a curve S in $\mathbb{T}$ is described in these coordinates by a pair of meromorphic functions (g, f) on a Riemann surface M, then, with respect to a certain choice of basis for $\mathrm{H}^0(\mathbb{P}_1, \mathcal{O}(\mathbb{T}))$ [11], the coordinate functions of the auxiliary null curve $\psi : M^* \longrightarrow \mathbb{C}^3$ are given by the Weierstrass formulae (1)–(3).

Now, $\mathcal{Q}_0 = \mathcal{C}(Q_1)$ fixes a point in $\mathbb{P}_9$, the space parameterising all quadrics in $\mathbb{P}_3$. Any other quadric $\mathcal{Q}_1$ intersects $\mathcal{Q}_0$ in a curve which completes a curve on $\mathbb{T}$ of the form

$$a\eta^2 + b(\zeta)\eta + c(\zeta) = 0, \tag{11}$$

where a is constant, $\deg(b) \le 2$ and $\deg(c) \le 4$. Thus the $\mathbb{P}_8$ of pencils which contain $\mathcal{Q}_0$ may be identified with the complete linear system $|2E_0|$ of such curves on $\mathbb{T}$.

Let $\Delta(\zeta) = b^2(\zeta) - 4ac(\zeta)$. Suppose that $a \neq 0$, and the branch locus $(\Delta(\zeta) = 0)$ comprises four distinct points (one of which may be at $\zeta = \infty$). In this generic case of transverse intersection, (11) is a smooth elliptic curve S on $\mathbb{T}$, double covering $\mathbb{P}_1$, where the projection map has four branch points. Completing the square, and letting $\mu = \eta + b(\zeta)/2a$, (11) becomes $\mu^2 = \Delta(\zeta)/4a^2$. The point being that the branch locus lies on the global section $\eta = -b(\zeta)/2a$. Of course, changing the choice of zero section merely translates the origin in $\mathrm{H}^0(\mathbb{P}_1, \mathcal{O}(\mathbb{T}))$.

By applying a fractional linear transformation we can move the four branch points on $\mathbb{P}_1$ to any other 4-tuple with the same cross-ratio. Thus the curve S may be brought to any one of the standard forms; cf. [2,7]. The corresponding bundle automorphism of $\mathbb{T}$ induces via the adjoint representation an $SO(3, \mathbb{C})$-rotation of $\mathrm{H}^0(\mathbb{P}_1, \mathcal{O}(\mathbb{T}))$ [10].

Remark. This can also be understood in terms of the four singular elements in the pencil of quadrics with base locus S, and projective equivalence of pencils; cf. Proposition 22.38 in [4].

This means we can reformulate Theorem 1 as follows:

Corollary 1. *Suppose that S is an elliptic curve in $|2E_0|$ as above. If $\Phi : S^* \longrightarrow \mathbb{C}^3$ is the auxiliary null curve induced by osculation then S admits a parameterisation $p : \mathbb{C}/\Lambda \longrightarrow S$, so that $\Phi \circ p = z + \lambda B \Omega$, for some $z \in \mathbb{C}^3$, $\lambda \in \mathbb{C}^*$ and $B \in SO(3, \mathbb{C})$, where Ω is as in (6).*

Remarks. 1. Theorem 1 shows that Ω's dual curve on $\mathbb{T}$ is an elliptic double cover of $\mathbb{P}_1$, branched at four distinct points on $\mathbb{P}_1$. This can be deduced by direct calculation of Ω's dual.

2. Much of the geometry of the null curve and the associated minimal surface can be read directly off S, without recourse to explicit formulae. See [9,10] for further details.

3. For real curves with rectangular lattices, $e_1 > e_2 > e_3$, and hence A in Theorem 1 lies in $SO(3, \mathbb{R})$. In this case we obtain branched minimal immersions of twice-punctured Klein bottles into $\mathbb{R}^3$. This is worked out in [11] for the special case of charge 2 monopole spectral curves.

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